4.8 – Row Space, Column Space, and Null Space

Definition: A homogeneous linear system $A\mathbf{x} = \mathbf{0}$ and an associated nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ are **corresponding linear systems**.

#28 Find a general solution of the system, and use that solution to find a general solution of the associated homogeneous system and a particular solution of the given system.

$$\begin{bmatrix} 9 & -3 & 5 & 6 \\ 6 & -2 & 3 & 1 \\ 3 & -1 & 3 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -8 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 9 & -3 & 5 & 6 & 4 \\ 6 & -2 & 3 & 1 & 5 \\ 3 & -1 & 3 & 14 & -8 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1/3 & 0 & -13/3 & 13/3 \\ 0 & 0 & 1 & 9 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = \tfrac{1}{3}x_2 + \tfrac{13}{3}x_4 + \tfrac{13}{3}, \quad x_3 = -9x_4 - 7$$

Let $s = x_2$, $t = x_4$

general solution to system

$$\vec{x} = \begin{bmatrix} \frac{1}{3}s + \frac{13}{3}t + \frac{13}{3} \\ s \\ -9t - 7 \\ t \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 13/3 \\ 0 \\ -9 \\ 1 \end{bmatrix} t + \begin{bmatrix} 13/3 \\ 0 \\ -7 \\ 0 \end{bmatrix}$$

This is here because the system is nonhomogeneous

This is a particular solution $\vec{x}_p$

The general solution to the associated homogeneous system is

$$\vec{x}_h = \begin{bmatrix} 1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 13/3 \\ 0 \\ -9 \\ 1 \end{bmatrix} t$$

Definition: The solution set of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is called the **general solution**. A constant vector $\mathbf{x}_0$ in the solution set of a consistent linear system $A\mathbf{x} = \mathbf{b}$ is a **particular solution**.

Theorem 4.8.2 If $\mathbf{x}_0$ is any solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$, and if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for the null space of A, then every solution of $A\mathbf{x} = \mathbf{b}$ can be expressed in the form $\mathbf{x} = \boxed{\mathbf{x}_0} + \boxed{c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_k \mathbf{v}_k}$. Conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$, the vector $\mathbf{x}$ in this formula is a solution of $A\mathbf{x} = \mathbf{b}$.

Note that the solution space of $A\vec{x} = \vec{b}$ can be considered a translation of the solution space of $A\vec{x} = \vec{0}$ by the particular solution vector.

The vectors in the general solution are linearly independent and so form a basis for the solution space of the homogeneous linear system, called the null space.
the solution is denoted $\vec{x}_h$

(The sol'n to the nohomogeneous system translates the null space.)

Definition: For an $m \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ the vectors

$$\begin{aligned} \mathbf{r}_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \mathbf{r}_2 &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ \vdots & \qquad\qquad \vdots \\ \mathbf{r}_m &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

in R^n that are formed from the rows of A are called the **row vectors** of A, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \ \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ \ldots, \ \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in R^m that are formed from the columns of A are called the **column vectors** of A.

Definition: If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is denoted by row (A) and is called the **row space** of A, and the subspace of R^m spanned by the column of vectors of A is denoted by col (A) and is called the **column space** of A. The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of R^n, is denoted by null (A) and is called the **null space** of A.

Theorem 4.8.1 A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Theorem 4.8.3
a) Row equivalent matrices have the same row space.
b) Row equivalent matrices have the same null space.

Theorem 4.8.4 If a matrix R is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.

In exercises 9 and 10, find bases for the null space and row space of A.

#9

b. $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$2x_1 = x_3 \Rightarrow x_1 = \frac{1}{2}x_3$

x_2 is free

A basis for row(A) is $\{(2, 0, -1)\}$.

A basis for the null space comes from solving the associated system:

$$\vec{x} = \begin{bmatrix} \frac{1}{2}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s$$

basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

#10

a. $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 7 & 7 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 7 & 0 & 7 & -2 \\ 0 & 7 & 7 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Basis for row space:

$\{(7,0,7,-2),\ (0,7,7,4)\}$

Basis for null space:

$x_1 = -x_3 + \frac{2}{7}x_4$

$x_2 = -x_3 - \frac{4}{7}x_4$

$$\vec{x} = \begin{bmatrix} -s + \frac{2}{7}t \\ -s - \frac{4}{7}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 2/7 \\ -4/7 \\ 0 \\ 1 \end{bmatrix} t$$

Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 7 \end{bmatrix} \right\}.$

Theorem 4.8.5 If A and B are row equivalent matrices, then:

a) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.

b) A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B. Row operations preserve the dimension of a column space

#15 Find a basis for the subspace of R^4 that is spanned by the given vectors.

$(1,\ 1,\ 0,\ 0),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 0,\ 2,\ 2),\ (0,\ -3,\ 0,\ 3)$

$$\overset{A}{\begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}} \rightarrow \overset{B}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

the given vectors form a basis for R^4

#17 Find a subset of the given vectors that forms a basis for the space spanned by those vectors, and then express each vector that is not in the basis as a linear combination of the basis vectors.

$\mathbf{v}_1 = (1,\ -1,\ 5,\ 2)$, $\mathbf{v}_2 = (-2,\ 3,\ 1,\ 0)$, $\mathbf{v}_3 = (4,\ -5,\ 9,\ 4)$, $\mathbf{v}_4 = (0,\ 4,\ 2,\ -3)$, $\mathbf{v}_5 = (-7,\ 18,\ 2,\ -8)$

$$\begin{array}{ccccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{array}$$
$$\begin{bmatrix} 1 & -2 & 4 & 0 & -7 \\ -1 & 3 & -5 & 4 & 18 \\ 5 & 1 & 9 & 2 & 2 \\ 2 & 0 & 4 & -3 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 8 & 15 \\ 0 & 1 & -1 & 4 & 11 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{array}{c} \begin{array}{ccccc} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 & \vec{w}_4 & \vec{w}_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Row echelon form gives leading 1s

Reduced row echelon form

Basis: $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$

Note that $\vec{w}_3 = 2\vec{w}_1 - \vec{w}_2$, $\vec{w}_5 = -\vec{w}_1 + 3\vec{w}_2 + 2\vec{w}_4$

What is $2\vec{v}_1 - \vec{v}_2$? $2\begin{bmatrix} 1 \\ -1 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 9 \\ 4 \end{bmatrix} = \vec{v}_3$

These are depency relationships among $\vec{w}_i$. It turns out these are preserved by elementary row operations.

$\vec{v}_3 = 2\vec{v}_1 - \vec{v}_2$, $\vec{v}_5 = -\vec{v}_1 + 3\vec{v}_2 + 2\vec{v}_4$

Definition: An equation that expresses a column vector of a matrix A that does not contain a pivot as a linear combination of column vectors that contain pivots is a **dependency equation**.

#18 Find a basis for the row space of A that consists entirely of row vectors of A.

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

Problematic approach: $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 7 & 0 & 7 & 2 \\ 0 & 7 & 7 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Basis for col(A) is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \right\}$.

~~Basis for row(A) is $\{(7,0,7,2),(0,7,7,4)\}$.~~

We use A^T: $\begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The desired basis is $\{(1,4,5,2),(2,1,3,0)\}$.